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GRUNSKY INEQUALITIES AND QUASICONFORMAL EXTENSION*

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In memory of Jürgen Moser

ABSTRACT

The Grunsky coefficient inequalities play a crucial role in various problems and are intrinsically connected with the integrable holomorphic quadratic differentials having only zeros of even order. For the functions with quasiconformal extensions, the Grunsky constant $\varkappa(f)$ and the extremal dilatation $k(f)$ are related by $\varkappa(f) \leq k(f)$. In 1985, Jürgen Moser conjectured that any univalent function $f(z) = z + b_0 + b_1 z^{-1} + \cdots$ on $\Delta^* = \{|z| > 1\}$ can be approximated locally uniformly by functions with $\varkappa(f) < k(f)$. In this paper, we prove a theorem confirming Moser's conjecture, which sheds new light on the features of Grunsky coefficients.

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1. Introduction and main theorem

1.1. How the Grunsky inequalities arise. We consider quasiconformal maps f of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ whose **Beltrami coefficients** $\mu_f = \partial_{\bar z} f/\partial_z f$ are supported in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and which have the expansion

$$f(z) = z + b_0 + b_1 z^{-1} + \cdots, \quad |z| > 1. \tag{1.1}$$

The deviation from conformality is estimated by means of the quantities $k(f) = \|\mu\|_\infty$ and $K(f) = (1 + k(f))/(1 - k(f)) > 1$, which are called the **dilatation** of and the **maximal dilatation** of a map f, respectively.

The conformal maps of the disk $\Delta^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ into $\widehat{\mathbb{C}} \setminus \{0\}$ with hydrodynamical normalization (1.1) form the well-known class Σ; its subclass of the maps with k- quasiconformal extensions to $\widehat{\mathbb{C}}$ is denoted by $\Sigma(k)$.

One defines for each $f \in \Sigma$ its **Grunsky coefficients** α_{mn} from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

where the branch of the logarithmic function is chosen which tends to zero as $z = \zeta \to \infty$.

The fundamental Grunsky univalence criterion says that a $\widehat{\mathbb{C}}$- holomorphic map (1.1) is univalent in Δ^* if and only if the inequalities

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn} x_m x_n \right| \le 1 \tag{1.2}$$

hold for all sequences $\mathbf{x} = (x_1, x_2, \dots)$ which are the points of the Hilbert space l^2 with norm $\|\mathbf{x}\| = (\sum_1^\infty |x_n|^2)^{1/2} = 1$ (cf. [Gr]). There exist certain other well-known equivalent inequalities of such kind ensuring the global univalence of locally univalent holomorphic functions in Δ^*.

It was established in [Ku1] that for the functions $f \in \Sigma(k)$ the inequality (1.2) is sharpened as follows:

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn} x_m x_n \right| \le k. \tag{1.3}$$

On the other hand, the important result of Pommerenke [Po] (Theorem 9.12), reproved by Zhuravlev using another method (see [Zh]; [KK], pp. 82–84), states that if $f \in \Sigma$ satisfies the inequality (1.3) for all indicated $\mathbf{x}$, then it admits a

k'- quasiconformal extension to $\widehat{\mathbb{C}}$ with $k' \geq k$. An explicit $k' = k'(k)$ is given in [Ku4].

A natural and important question posed by various authors (see, e.g., [B], [Ku1], [Le1]) is whether the inequalities (1.3) ensure a k- quasiconformal extension of $f \in \Sigma$ with the same k.

A characterization of the functions for which the inequality (1.3) is both necessary and sufficient to belong to $\Sigma(k)$ was given in [Kr2], [Kr3]; in [Ku3] more explicitly for functions which transform $|z| = 1$ onto an analytic Jordan curve. To present it, we need some notations. We call the quantity

$$\varkappa(f) = \sup \left\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\alpha_{mn}x_m x_n \Big| : \sum_{1}^{\infty} |x_n|^2 = 1 \right\}$$

the **Grunsky constant** of a function $f \in \Sigma$ and denote by $k(f)$ the minimal dilatation among all quasiconformal extensions of f to $\widehat{\mathbb{C}}$.

Let $A_1(\Delta)$ denote the subspace of $L_1(\Delta)$ formed by holomorphic functions in Δ, and let

$$A_1^2 = \{\psi \in A_1(\Delta) : \psi = \varphi^2\};$$

this set consists of integrable holomorphic functions in Δ having only zeros of even order. Put $< \mu, \psi >_\Delta = \int_\Delta \mu\psi dm_2$ for $\mu \in L_\infty(\Delta)$ and $\psi \in L_1(\Delta)$, where m_2 is the Lebesgue measure on $\mathbb{C}$.

The indicated result of [Kr2], [Kr3] says that *the equality*

$$\varkappa(f) = k(f) \tag{1.4}$$

holds if and only if the function f is the restriction to Δ^* of a quasiconformal homeomorphism w^{μ_0} of $\widehat{\mathbb{C}}$ whose Beltrami coefficient μ_0 satisfies the equality

$$\sup | < \mu_0, \varphi >_\Delta | = \|\mu_0\|_\infty, \tag{1.5}$$

where the supremum is taken over holomorphic functions $\varphi \in A_1^2$ with $\|\varphi\|_{A_1(\Delta)} = 1$.

This result reveals a deep connection between the Grunsky coefficients and integrable holomorphic quadratic differentials with zeros of even order. Such differentials play a crucial role in various problems, in particular, in applications to Teichmüller spaces (see, e.g., [Kr4]).

It turned out that for a wide class of the boundary curves $f(\partial\Delta)$ the equality (1.4) is both nesessary and sufficient for the existence of the Teichmüller–Kühnau extension of a function $f \in \Sigma$, i.e., with the Beltrami coefficient of the form $\mu_f = k|\varphi|/\varphi$ with $\varphi \in A_1^2$ (cf., e.g., [Kr3], [Ku3]).

There is a geometric interpretation of these results in terms of invariant metrics on the universal Teichmüller space (see the next section).

1.2. MOSER'S CONJECTURE AND MAIN RESULT. In September 1985 Jürgen Moser, discussing the talk of the second author on the XI. Österreichischer Mathematikerkongress in Graz, had conjectured that the set of the functions satisfying (1.4) must be rather sparse and the functions with $\varkappa(f) < k(f)$ approximate all functions of Σ uniformly on compact sets in Δ^*.

In this paper, we prove the theorem confirming Moser's conjecture, which sheds new light on the features of Grunsky coefficients.

THEOREM 1.1: *For every function $f \in \Sigma$, there exists a sequence of functions $f_n \in \bigcup_k \Sigma(k)$ with $\varkappa(f_n) < k(f_n)$ for all n, which is uniformly convergent to f on compact sets in Δ^*.*

We give here two alternate proofs of Theorem 1.1.

1.3. REMARKS. The convergence of the maps f_n to f yields that their Schwarzian derivatives

$$S_{f_n} = \left(\frac{f_n''}{f_n'}\right)' - \frac{1}{2}\left(\frac{f_n''}{f_n'}\right)^2$$

are convergent to S_f uniformly on compact sets in Δ^*.

On the other hand, the derivatives S_f of $f \in \bigcup_k \Sigma(k)$ fill a bounded open domain in the space **B** of holomorphic functions in Δ^* with the norm $\|\varphi\| = \sup_{\Delta^*}(|z|^2 - 1)^2|\varphi(z)|$. This domain models the universal Teichmüller space **T**. It would be interesting to know how sparse in **T** is the set of points S_f satisfying (1.4).

Another important question for applications is to provide the sufficient conditions for the functions $f \in \Sigma$ ensuring the existence of k- quasiconformal extensions with an explicit $k = k(\varkappa(f))$. Such conditions have been given in [Ku4], [Kr5].

Finally, let us formulate, in contrast to Theorem 1.1, the following

CONJECTURE: *For every function $f \in \Sigma$ with $\varkappa(f) < k(f)$, there does not exist a sequence of functions $f_n \in \bigcup_k \Sigma(k)$ with $\varkappa(f_n) = k(f_n)$, which is uniformly convergent to f on compact sets in Δ^*.*

2. First proof of Theorem 1.1 (first author)

The proof consists of two steps.

STEP 1: Quasiconformal deformations of Fuchsian groups.

THEOREM 2.1: *Let Γ be a torsion free finitely generated Fuchsian group of the first kind acting on Δ (so that the orbit space Δ/Γ is a hyperbolic Riemann surface of finite analytic type). Then every extremal Beltrami differential*

$$\mu_0 \in \mathrm{Belt}(\Delta,\Gamma)_1 := \{\mu \in L_\infty(\mathbb{C}) : \mu|\Delta^* = 0, \|\mu\| < 1; (\mu\circ\gamma)\overline{\gamma'}/\gamma' = \mu \text{ for } \gamma \in \Gamma\}$$

determines a quasiconformal homeomorphism f^{μ_0} of $\widehat{\mathbb{C}}$ compatible with the group Γ and such that

$$\varkappa(f^{\mu_0}) < k(f^{\mu_0}).$$

Proof: As is well-known, the Teichmüller space $\mathbf{T}(\Gamma)$ of the group Γ is embedded into $\mathbf{T}$, and $\mathbf{T}(\Gamma) = \mathbf{T} \cap \mathbf{B}(\Gamma)$, where $\mathbf{B}(\Gamma)$ is the subspace of $\mathbf{B}$ formed by Γ-automorphic forms φ of the weight -4 (quadratic differentials) in Δ^*, i.e., $(\varphi\circ\gamma)(\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$ (see, e.g., [Le2]).

Denote by $A_1(\Delta,\Gamma)$ the space of holomorphic Γ-automorphic forms of the weight -4 in Δ integrable over Δ/Γ, with the norm $\|\psi\| = \int_{\Delta/\Gamma} |\psi| dm_2$. For every $\mu \in \mathrm{Belt}(\Delta,\Gamma)_1$, we have the pairings

$$<\mu,\psi>_{\Delta/\Gamma} = \int_{\Delta/\Gamma} \mu\psi dm_2, \quad <\mu,\Psi>_\Delta = \int_\Delta \mu\Psi dm_2$$

with $\psi \in L_1(\Delta/\Gamma)$ and $\Psi \in L_1(\Delta)$, respectively.

It is well-known (cf. [Be]) that the Poincaré theta-operator

$$\Theta_\Gamma\Psi(z) = \sum_{\gamma\in\Gamma} \Psi(\gamma z)\gamma'(z)^2 : A_1(\Delta) \to A_1(\Delta,\Gamma)$$

acts on $A(\Delta)$ surjectively, and by McMullen's theorem [MM] its norm

$$\|\Theta_\Gamma\| < 1. \tag{2.1}$$

Any extremal Beltrami differential $\mu_0 \in \mathrm{Belt}(\Delta,\Gamma)_1$ is of Teichmüller form

$$\mu_0 = \|\mu_0\|_\infty \frac{|\psi_0|}{\psi_0} \quad \text{with } \psi_0 \in A_1(\Delta,\Gamma), \|\psi_0\| = 1,$$

and we have equality

$$\begin{aligned} \|\mu_0\|_\infty &= \max\{|<\mu_0,\psi>_{\Delta/G}| : \|\psi_0\|_{A_1(\Delta,\Gamma)} = 1\} \\ &= |<\mu_0,\psi_0>_{\Delta/G}| = |<\mu_0,\Psi_0>_\Delta| \end{aligned} \tag{2.2}$$

for any Ψ_0 with $\Theta_\Gamma\Psi_0 = \psi_0$.

Let us now compare the value $\|\mu_0\|_\infty$ with the minimal dilatation in the class $[f^{\mu_0}]_{\partial\Delta}$ of all quasiconformal extensions of $f^{\mu_0}|\partial\Delta$ into the disk Δ, i.e., with the quantity $k(f^{\mu_0})$.

By the Hamilton–Krushkal–Reich–Strebel theorem, a Beltrami coefficient $\mu \in \mathrm{Belt}(\Delta)_1$ is extremal (i.e., $\|\mu\|_\infty = \inf\{\|\nu\|_\infty : f^\nu \in [f^\mu]_{\partial D}\}$) if and only if

$$\|\mu\|_\infty = \sup\{| < \mu, \Psi >_\Delta | : \|\Psi\|_{A_1(\Delta)} = 1\} \tag{2.3}$$

(see, e.g., [GL], [Ha], [Kr1], [RS]).

Thus we have to estimate $< \mu_0, \Psi >_\Delta$ on the sphere $\{\|\Psi\|_{A_1(\Delta)} = 1\}$. In view of the equality $< \mu_0, \Psi >_\Delta = < \mu_0, \Theta_\Gamma \Psi >_{\Delta/\Gamma}$, we can ignore the elements $\Psi \in \ker \Theta_\Gamma$.

Consider for $\psi \in A_1(\Delta, \Gamma)$ with $\|\psi\| = 1$ the set

$$D(\psi) = \{\Psi \in A_1(\Delta) : \Theta_\Gamma \Psi = \psi\},$$

noting that the differences $\Psi_1 - \Psi_2$ of any two elements of $D(\psi)$ belong to $\ker \Theta_\Gamma$. Let

$$\sigma(\psi) := \inf\{\|\Psi\|_{A_1(D)} : \Psi \in D(\psi)\}.$$

Using the inequality (2.1), one concludes that

$$\sigma(\psi) \geq \frac{1}{\|\Theta_\Gamma\|} > 1. \tag{2.4}$$

Now fix sufficiently small $\varepsilon > 0$ and take $\widetilde{\Psi} \in D(\psi)$ so that $\|\widetilde{\Psi}\|_{A_1(\Delta)} = \sigma(\psi) + \varepsilon$. Put

$$\Psi = \frac{\widetilde{\Psi}}{\sigma(\psi)}, \quad \Psi_\varepsilon = \frac{\widetilde{\Psi}}{\sigma(\psi) + \varepsilon}.$$

Then, using (2.4), we get

$$\begin{aligned} | < \mu_0, \Psi_\varepsilon >_\Delta | &= | < \mu_0, \Psi >_\Delta | + O(\varepsilon) = \frac{1}{\sigma(\psi) + \varepsilon} | < \mu_0, \psi >_{\Delta/\Gamma} | \\ &\leq \frac{1}{\sigma(\psi) + \varepsilon} \|\mu_0\|_\infty \leq \|\Theta_\Gamma\| \|\mu_0\|_\infty + O(\varepsilon). \end{aligned} \tag{2.5}$$

As $\varepsilon \to 0$, one obtains that the functions $\Psi \in A_1(\Delta)$ with $\|\Psi\| = 1$ satisfy the inequality

$$| < \mu_0, \Psi >_\Delta | \leq \|\Theta_\Gamma\| \|\mu_0\|_\infty. \tag{2.6}$$

In view of (2.1) and (2.3), inequality (2.6) implies that the Beltrami coefficient μ_0 cannot be extremal in the class, $[f^{\mu_0}]_{\partial\Delta}$; and therefore, for any extremal Beltrami coefficient ν_0 in this class we have

$$(2.7)\quad \|\nu_0\|_\infty = \sup\{| < \nu_0, \Psi >_\Delta | : \Psi \in A_1(\Delta), \|\Psi\| = 1\} = k(f^{\mu_0}) < \|\mu_0\|_\infty.$$

Put $\mu_0^* = \mu_0/\|\mu_0\|_\infty$ and consider in the space $\mathbf{T}(\Gamma) \subset \mathbf{T}$ the holomorphic disk

$$(2.8)\qquad \Delta(\mu_0^*) = \{\phi_{\mathbf{T}(\Gamma)}(t\mu_0^*) : t \in \Delta\},$$

where $\phi_{\mathbf{T}(\Gamma)}$ is the holomorphic projection

$$\mu \to S_{f^\mu}\colon \operatorname{Belt}(\Delta, \Gamma)_1 \to \mathbf{T}(\Gamma) \subset \mathbf{B}(\Gamma).$$

The same map $\mu \to S_{f^\mu}$ induces the defining projection $\phi_{\mathbf{T}}\colon \operatorname{Belt}(\Delta)_1 \to \mathbf{T}$ (where $\operatorname{Belt}(\Delta)_1$ is the unit ball of all Beltrami coefficients supported in Δ).

The tangent vector to the disk (2.8) in $\mathbf{T}(\Gamma)$ at the origin is $\phi'_{\mathbf{T}(\Gamma)}(\mathbf{0})\mu_0^*$, and by (2.2) its Teichmüller norm is equal to one. On the other hand, the tangent vector to this disk in $\mathbf{T}$ is $\phi'_{\mathbf{T}}(\mathbf{0})\mu_0^*$ and has Teichmüller norm given by (2.7), i.e.,

$$(2.9)\qquad \|\phi'_{\mathbf{T}}(\mathbf{0})\mu_0^*\| = \|\nu_0\|_\infty/\|\mu_0\|_\infty < \mathbf{1}.$$

The inequality (2.9) allows us to conclude that

$$\varkappa(f^{t\mu_0^*}) < k(f^{t\mu_0^*}) \quad \text{for all } t \in \Delta \setminus \{0\}.$$

Indeed, as was shown in [Kr3], the equality $\varkappa(f^{t\mu_0^*}) = k(f^{t\mu_0^*})$, even for one $t \neq 0$, implies that the restrictions of the Carathéodory and Teichmüller metrics of the space $\mathbf{T}$ onto the disk $\Delta(\mu_0^*)$ must coincide, and therefore, this disk is extremal (geodesic in both metrics).

On the other hand, due to results of [EKK] and [Kr4], a holomorphic disk $h(\Delta) \subset \mathbf{T}$ is extremal if and only if the tangent vector $h'(0)$ has the Teichmüller length 1. This contradiction proves Theorem 2.1. ∎

Step 2. Approximation: We now apply an approximation of univalent functions on Δ^* making use of a construction close to [KG].

We take the set of points

$$E = \{e^{\pi m i/2^n}, m = 0, 1, \ldots, 2^{n+1} - 1; n = 1, 2, \ldots\}$$

on the unit circle and consider the punctured spheres

$$X_n = \widehat{\mathbb{C}} \setminus \{e^{\pi mi/2^n}, m = 0, 1, \dots, 2^{n+1} - 1\}$$

and their universal holomorphic covering maps $g_n\colon \Delta \to X_n$ normalized by $g_n(0) = 0, g_n'(0) > 0, n = 1, 2, \dots$.

The radial slits from the infinite point to all the points $e^{\pi mi/2^n}$ form a canonical dissection L_n of X_n and define the simply connected surface $X_n' = X_n \setminus L_n$. The covering map g_n determines a Fuchsian group Γ_n in the disk Δ of covering transformations. The (open) fundamental polygon P_n of Γ_n in Δ corresponding to dissection L_n is a regular circular 2^{n+1}-gon centered at the origin which can be chosen to have a vertex at 1. The restriction of g_n to P_n is univalent.

By the Carathéodory theorem on convergence of domains to a kernel (which in our case is the disk Δ), we obtain that both sequences $\{g_n|P_n\}$ and $\{g_n\}$ are convergent locally uniformly on Δ to the identity map.

We may now complete the proof of Theorem 1.1. It suffices to consider the functions admitting quasiconformal extensions, i.e., $f^\mu \in \bigcup_k \Sigma(k)$.

Given such a map f^μ, we take its k-quasiconformal extension $\widetilde{f^\mu}$ and approximate it by homeomorphisms $\widetilde{f^{\mu_n}}$ with the Beltrami differentials

$$\mu_n = (\mu_f \circ g_n)\overline{g_n'}/g_n', \quad n = 1, 2, \dots.$$

Each $\widetilde{f^{\mu_n}}$ is again k-quasiconformal and compatible with the group Γ_n. The differentials μ_n are convergent to μ almost everywhere on $\mathbb{C}$, thus the maps $\widetilde{f^{\mu_n}}$ are convergent to f^μ uniformly in the spherical metric on $\widehat{\mathbb{C}}$, which yields the assertion of Theorem 1.1.

3. Second proof of Theorem 1.1 (second author)

The idea is to approximate a given fixed $f \in \Sigma$ by such mappings of the class Σ which have an extremal quasiconformal extension of Teichmüller type with at least one single zero (near the unit circle) of the corresponding quadratic differential. For such mappings, see [Ku3] (Satz 3), the desired property $\varkappa < k$ occurs.

For a fixed $f \in \Sigma$ and a fixed $r > 1$, there exists by the Strebel frame criterion [St] a uniquely determined extremal quasiconformal extension (of Teichmüller type) of the values of $f(|z| = r)$ to the disk $|z| < r$. Let $\varphi_r(z)dz^2$ be the corresponding quadratic differential in the z-plane, $k_r(< 1)$ the corresponding (constant) dilatation, $\mu_r(z) = k_r|\varphi_r|/\varphi_r$ (for $|z| < r$) the Beltrami coefficient. (Finally we will choose $r - 1$ small, such that $\frac{1}{r}f(rz) \in \Sigma$ is close to $f(z)$.)

We choose now $r' < r$ such that $r - r'$ is small and such that φ_r does not vanish at $|z| = r'$. With the modified Beltrami coefficient

$$\widetilde{\mu}_{r,r'}(z) = \begin{cases} \mu_r(z) & \text{for } |z| < r', \\ 0 & \text{for } |z| > r, \end{cases}$$

we obtain a new hydrodynamically normalized mapping $\widetilde{f}$ which is outside a great circle close to $f(z)$. For, this modification (only in the small ring $r' < |z| < r$) of the Beltrami coefficient yields only a small modification of the mapping, because we have in the image planes conformality outside a small area (cf. [KK], Zweiter Teil, Kap. IV, (17)).

Finally, we choose, with a small $\delta > 0$, the following further modified Beltrami coefficient:

$$\mu^* = \mu^*_{r,r',\delta}(z) = k_r \frac{|\varphi_r|}{\varphi_r} \frac{|z - N|}{z - N} \frac{z - P}{|z - P|}, \quad |z| \leq r', \tag{3.1}$$

with $N = r' - \delta$, $P = r' + \delta$, that means the modified quadratic differential

$$\varphi^* dz^2 = \varphi_r \frac{z - N}{z - P} dz^2. \tag{3.2}$$

We have

$$|\mu^* - \tilde{\mu}| = k_r \left| \frac{|z - N|}{z - N} \frac{z - P}{|z - P|} - 1 \right|. \tag{3.3}$$

For elementary geometric reasons, this is arbitrarily small for $\{|z| < r'\} \cap \{|z - r'| > d\}$ if we choose first a small $d > 0$ and then a small $\delta < d$.

If we define furthermore $\mu^* \equiv 0$ for $|z| > r'$, we get with this new Beltrami coefficient μ^*, now defined in the whole plane, a new hydrodynamically normalized mapping f^*. The part of this mapping for $|z| \leq r'$ is extremal for the corresponding boundary values, because the quadratic differential is analytic for $|z| \leq r'$. And because at $|z| = r'$ there are no zeros, we obtain a closed analytic Jordan curve as the image of $|z| = r'$. Now this new quadratic differential has a single zero at N with $|N| < r'$. Therefore we have for this mapping f^* (better to say: for $f^{**} = f^*(r'z)/r')$) the strong inequality $\varkappa < k(= k_r)$, see [Ku3] (Satz 3). And the function f^* approximates f because for the mapping between the arising image planes the Beltrami coefficient is small (cf. (3.3)) outside a small area, namely, outside the image of $|z - r'| < d$ (fixed r', then small d, then small $\delta < d$). This follows again with the mentioned formula (17) in [KK].

Therefore, with these functions f^* we can construct a sequence of mappings $\in \Sigma$, converging to f and with the property $\varkappa < k$, desired in Theorem 1.1.

This proof immediately also yields the following

COROLLARY 3.1: *The functions f_n in Theorem 1.1 can be chosen in such a way that the images of $|z| = 1$ are closed analytic Jordan curves.*

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